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FIXED POINT THEOREMS FOR THE EXTENSIONS OF PANDHARE AND WAGHMODE IN HILBERT SPACE

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Abstract. The purpose of this paper is to present common fixed point theorems in Hilbert space which are the generalization of well known proved results. The results which considered here are extended for a pair of mappings and also some positive integer's powers of them involving more rational terms in the inequality. Furthermore we developed these results to a sequence of mappings on the same space.

Key words: Hilbert space, closed subset, Cauchy sequence, completeness

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1. Introduction

In mathematics, fixed point theory plays a vital role in abroad set of applications in pure and applied mathematics. The contraction is the main tool among all to provide the existence and uniqueness of the fixed point in theoretical mathematics. The research in the discipline of fixed point theory and approximation theory was initiated by Banach in 1922, since then many generalizations have been taken into considerations of using contractive conditions or imposing some additional conditions on different spaces. The growth of this area of knowledge have been extensively reported by many authors in the treatises/ research papers of [2, 5, 7, 8, 12, 13, 14, 15, 16, 26, 27, 29, 31, 34] in metric space by involving either ordinary or rational terms in the inequalities presented there. After that some have got refined or generalized by authors [3, 4, 18, 20, 21, 23, 25] in different spaces. The existence and uniqueness of a fixed point for a pair of self / different continuous mappings on various spaces can be found in [1, 6, 9, 22, 32]. The common fixed point for a sequence of continues or non-continues self or different mappings imposing some weaker conditions on the mappings as well as over the spaces can see in [10, 11, 17, 19, 24, 28, 32].

Later this celebrated principle has been extended and generalized by several researchers in various spaces such as quasi-metric spaces, cone metric spaces, G-metric spaces, partial metric spaces, vector valued metric spaces and normed species etc. The generalized fixed point's properties of family of mappings in Hilbert spaces can be found in [5, 6, 7, 32]. Some generalization of Banach fixed point theorems in Hilbert spaces are as follows:

i) Koparde and Wghmode [17] have proved fixed point theorem for a self-mapping on a closed subset of Hilbert space, satisfying the Kannan type condition

$$\|Tx-Ty\|^2 \le \alpha\left\{\|x-Tx\|^2+\|y-Ty\|^2\right\}$$

for all $x, y \in X$ with $x \neq y$ and $\alpha \in [0,1/2)$.

ii) Sharma et.al [30] have proved the following a common fixed point theorem for self mapping satisfying the following condition

$$\|Tx-Ty\| \le \alpha \frac{\|x-Tx\|^2+\|y-Ty\|^2}{\|x-Tx\|+\|y-Ty\|}+\beta\|x-y\|$$

for all $x, y \in X$ with $x \neq y$ and $0 \le \alpha < 1/2, 0 \le \beta, 2\alpha+\beta<1$.

In this manuscript, we prove that an operator T satisfying certain rational contraction condition has a unique fixed point on a closed subset X of Hilbert space and then generalized the same result to a pair of mappings T_1, T_2 , some positive integers powers p , q of a pair mappings T_1^p, T_2^q and then further extended to a sequence of mappings. In all cases we have obtained a common fixed point in X . Our results generalize the main result of Koparde and Wghmode [17] and Pandhare and Waghmode [23].

2. Main Results

Theorem: 1 Let X be a closed subset of a Hilbert space and $T: X \to X$ be a self mapping satisfying the following inequality

$$\|Tx-Ty\|^2 \le a_1 \frac{\|x-Tx\|^2\left[1+\|y-Ty\|^2\right]}{1+\|x-y\|^2}+a_2 \frac{\|y-Ty\|^2\left[1+\|y-Tx\|^2\right]}{1+\|x-y\|^2}$$

$$+a_3 \frac{\|x-Ty\|^2\left[1+\|y-Tx\|^2\right]}{1+\|x-y\|^2}+a_4 \frac{\|x-y\|^2\left[1+\|Tx-Ty\|^2\right]}{1+\|x-y\|^2}$$

$$+a_5 \frac{\|x-Ty\|^2\left[1+\|Tx-Ty\|^2\right]}{1+\|x-y\|^2}+a_6 \frac{\|x-y\|^2\left[1+\|x-Ty\|^2\right]}{1+\|x-y\|^2}$$

$$+a_7 \frac{\|x-Tx\|^2+\|y-Ty\|^2+\|x-y\|^2}{1+\|x-Tx\|^2\|x-Ty\|^2\|y-Tx\|^2\|x-y\|^2}+a_8\|x-y\|^2$$

for all $x, y \in X$ with $x \neq y$ and $a_i (i = 1, 2, 3, ..., 8)$ are non-negative real's with $a_1 + 2a_2 + a_3 + a_4 + a_5 + a_6 + 3a_7 + a_8 < 1$. Then T has a unique fixed point in X.

Proof: For any $x_0 \in X$, we define a sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n, \text{ for } n = 0, 1, 2, 3, ...$$

Now, we claim that the sequence $\{x_n\}$ is a Cauchy sequence in X. For this consider

$$\|x_{n+1} - x_n\|^2 = \|Tx_n - Tx_{n-1}\|^2$$

Then by the hypothesis, we have

$$\|x_{n+1} - x_n\|^2 \leq a_1 \frac{\|x_n - Tx_n\|^2 \left[1 + \|x_{n-1} - Tx_{n-1}\|^2\right]}{1 + \|x_n - x_{n-1}\|^2} + a_2 \frac{\|x_{n-1} - Tx_{n-1}\|^2 \left[1 + \|x_{n-1} - Tx_n\|^2\right]}{1 + \|x_n - x_{n-1}\|^2}$$

$$+ a_3 \frac{\|x_n - Tx_{n-1}\|^2 \left[1 + \|x_{n-1} - Tx_n\|^2\right]}{1 + \|x_n - x_{n-1}\|^2} + a_4 \frac{\|x_n - x_{n-1}\|^2 \left[1 + \|Tx_n - Tx_{n-1}\|^2\right]}{1 + \|x_n - x_{n-1}\|^2}$$

$$+ a_5 \frac{\|x_n - Tx_{n-1}\|^2 \left[1 + \|Tx_n - Tx_{n-1}\|^2\right]}{1 + \|x_n - x_{n-1}\|^2} + a_6 \frac{\|x_n - x_{n-1}\|^2 \left[1 + \|x_n - Tx_{n-1}\|^2\right]}{1 + \|x_n - x_{n-1}\|^2}$$

$$+ a_7 \frac{\|x_n - Tx_n\|^2 + \|x_{n-1} - Tx_{n-1}\|^2 + \|x_n - x_{n-1}\|^2}{1 + \|x_n - Tx_n\|^2 \|x_n - Tx_{n-1}\|^2 \|x_{n-1} - Tx_n\|^2 \|x_n - x_{n-1}\|^2} + a_8 \|x_n - x_{n-1}\|^2$$

$$\Rightarrow \quad \|x_{n+1} - x_n\|^2 \leq s(n) \|x_n - x_{n-1}\|^2$$

wherein $s(n) = \dfrac{(a_2 + a_4 + a_6 + 2a_7 + a_8) + (a_2 + 2a_7 + a_8)\|x_n - x_{n-1}\|^2}{(1 - a_1 - a_7) + (1 - a_1 - a_2 - a_4 - a_7)\|x_n - x_{n-1}\|^2}$, for $n = 1, 2, 3, ...$

It is clear that $S = s(n) < 1$, for all n as $a_1 + 2a_2 + a_3 + a_4 + a_5 + a_6 + 3a_7 + a_8 < 1$. Using the inequality from the hypothesis successively, we get

$$\|x_{n+1}-x_n\|^2 \le S^n \|x_1-x_0\|^2$$

Taking the limit to this inequality, we find that $\|x_{n+1}-x_n\| \to 0$, which shows that the sequence $\{x_n\}$ is a Cauchy sequence in X . Using the completeness of X , we can find a point $\mu \in X$ such that $x_n \to \mu$ as $n \to \infty$.

Consequently $\{x_{n+1}\}=\{Tx_n\}$ is a subsequence of $\{x_n\}$ and hence has the same limit μ . Since T is continuous, we obtain

$$T(\mu)=T(\lim_{n\to\infty} x_n)=\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = \mu$$

Hence μ is a fixed point of T in X . Now, it remains to show that μ is a unique fixed point of T . For this let v $(\mu \ne v)$ be another fixed point of T . Then

$$\|\mu-v\|^2 = \|T\mu - Tv\|^2$$

$$\le a_1 \frac{\|\mu-T\mu\|^2\left[1+\|v-Tv\|^2\right]}{1+\|\mu-v\|^2} + a_2 \frac{\|v-Tv\|^2\left[1+\|v-T\mu\|^2\right]}{1+\|\mu-v\|^2}$$

$$+a_3 \frac{\|\mu-Tv\|^2\left[1+\|v-T\mu\|^2\right]}{1+\|\mu-v\|^2} + a_4 \frac{\|\mu-v\|^2\left[1+\|T\mu-Tv\|^2\right]}{1+\|\mu-v\|^2}$$

$$+a_5 \frac{\|\mu-Tv\|^2\left[1+\|T\mu-Tv\|^2\right]}{1+\|\mu-v\|^2} + a_6 \frac{\|\mu-v\|^2\left[1+\|\mu-Tv\|^2\right]}{1+\|\mu-v\|^2}$$

$$+a_7 \frac{\|\mu-T\mu\|^2+\|v-Tv\|^2+\|\mu-v\|^2}{1+\|\mu-T\mu\|^2\|\mu-Tv\|^2\|v-T\mu\|^2\|\mu-v\|^2} + a_8\|\mu-v\|^2$$

which implies that

$$\|\mu-v\|^2 \le \left(a_3+a_4+a_5+a_6+a_7+a_8\right)\|\mu-v\|^2$$

which is a contradiction; for $a_3+a_4+a_5+a_6+a_7+a_8<1$. Therefore $\mu=v$ and hence μ is a unique fixed point of T in X .

This completes the proof of the theorem.

The following theorem is an extension of the above theorem involving two self mapping on a closed subset of a Hilbert space in the contraction condition.

Theorem: 2 Let T_1, T_2 be two self mappings on a closed subset X of a Hilbert space satisfying the following condition, then T_1 and T_2 have a unique common fixed point in X .

$$\|T_1x - T_2y\|^2 \le a_1 \frac{\|x - T_1x\|^2\left[1 + \|y - T_2y\|^2\right]}{1 + \|x - y\|^2} + a_2 \frac{\|y - T_2y\|^2\left[1 + \|y - T_1x\|^2\right]}{1 + \|x - y\|^2}$$

$$+ a_3 \frac{\|x - T_2y\|^2\left[1 + \|y - T_1x\|^2\right]}{1 + \|x - y\|^2} + a_4 \frac{\|x - y\|^2\left[1 + \|T_1x - T_2y\|^2\right]}{1 + \|x - y\|^2}$$

$$+ a_5 \frac{\|x - T_2y\|^2\left[1 + \|T_1x - T_2y\|^2\right]}{1 + \|x - y\|^2} + a_6 \frac{\|x - y\|^2\left[1 + \|x - T_2y\|^2\right]}{1 + \|x - y\|^2}$$

$$+ a_7 \frac{\|x - T_1x\|^2 + \|y - T_2y\|^2 + \|x - y\|^2}{1 + \|x - T_1x\|^2\,\|x - T_2y\|^2\,\|y - T_1x\|^2\,\|x - y\|^2} + a_8\|x - y\|^2$$

for all $x, y \in X$ with $x \ne y$ and $a_i (i = 1, 2, 3, ..., 8)$ are non-negative real's with $a_1 + a_2 + 2a_3 + a_4 + 2(a_5 + a_6) + 3a_7 + a_8 < 1$.

Proof : Let us construct a sequence $\{x_n\}$ for an arbitrary point $x_0 \in X$ as follows

$$x_{2n+1} = T_1x_{2n} \ , \ x_{2n+2} = T_2x_{2n+1} \text{ , for } n = 0,1,2,3,........$$

For examine the Cauchy sequence nature of $\{x_n\}$ in X consider the following and by hypothesis, we have

$$\|x_{2n+1} - x_{2n}\|^2 = \|T_1x_{2n} - T_2x_{2n-1}\|^2$$

$$\le a_1 \frac{\|x_{2n} - T_1x_{2n}\|^2\left[1 + \|x_{2n-1} - T_2x_{2n-1}\|^2\right]}{1 + \|x_{2n} - x_{2n-1}\|^2} + a_2 \frac{\|x_{2n-1} - T_2x_{2n-1}\|^2\left[1 + \|x_{2n-1} - T_1x_{2n}\|^2\right]}{1 + \|x_{2n} - x_{2n-1}\|^2}$$

$$+ a_3 \frac{\|x_{2n} - T_2x_{2n-1}\|^2\left[1 + \|x_{2n-1} - T_1x_{2n}\|^2\right]}{1 + \|x_{2n} - x_{2n-1}\|^2} + a_4 \frac{\|x_{2n} - x_{2n-1}\|^2\left[1 + \|T_1x_{2n} - T_2x_{2n-1}\|^2\right]}{1 + \|x_{2n} - x_{2n-1}\|^2}$$

$$+ a_5 \frac{\|x_{2n} - T_2x_{2n-1}\|^2\left[1 + \|T_1x_{2n} - T_2x_{2n-1}\|^2\right]}{1 + \|x_{2n} - x_{2n-1}\|^2} + a_6 \frac{\|x_{2n} - x_{2n-1}\|^2\left[1 + \|x_{2n} - T_2x_{2n-1}\|^2\right]}{1 + \|x_{2n} - x_{2n-1}\|^2}$$

$$+a_7 \frac{\|x_{2n}-T_1x_{2n}\|^2+\|x_{2n-1}-T_2x_{2n-1}\|^2+\|x_{2n}-x_{2n-1}\|^2}{1+\|x_{2n}-T_1x_{2n}\|^2\|x_{2n}-T_2x_{2n-1}\|^2\|x_{2n-1}-T_1x_{2n}\|^2\|x_{2n}-x_{2n-1}\|^2}+a_8\|x_{2n}-x_{2n-1}\|^2$$

which suggests that

$$\|x_{2n+1}-x_{2n}\|^2 \le s(n)\,\|x_{2n}-x_{2n-1}\|^2$$

where $s(n)=\dfrac{(a_2+a_4+a_6+2a_7+a_8)+(a_2+2a_7+a_8)\|x_{2n}-x_{2n-1}\|^2}{(1-a_1-a_7)+(1-a_1-a_2-a_4-a_7)\|x_{2n}-x_{2n-1}\|^2}$

A similar calculation gives that

$$\|x_{2n+2}-x_{2n+1}\|^2 \le t(n)\,\|x_{2n+1}-x_{2n}\|^2$$

where $t(n)=\dfrac{(a_1+a_3+a_4+a_5+a_6+2a_7+a_8)+(a_6+2a_7)\|x_{2n+1}-x_{2n}\|^2}{(1-a_2-a_3-a_5-a_7)+(1-a_1-a_4-a_6-a_7)\|x_{2n+1}-x_{2n}\|^2}$

Here both $s(n)$ and $t(n)$ depends n, since $a_1+a_2+2a_3+a_4+2(a_5+a_6)+3a_7+a_8<1$, we see that $s(n)<1$ and $t(n)<1$, for all n.

Let $S=\max\{s(n):n=0,1,2,......\}, T=\max\{t(n):n=0,1,2,......\}$, and let $\lambda^2=\max\{S,T\}$ so that $0<\lambda<1$, as result of which in general, we get

$$\|x_{n+1}-x_n\| \le \lambda\,\|x_n-x_{n-1}\|$$

Repeating the above process in a similar manner, we get

$$\|x_{n+1}-x_n\| \le \lambda^n\|x_1-x_0\|\ , n\ge 1$$

On taking $n\to\infty$, we obtain $\|x_{n+1}-x_n\|\to 0$. Hence it follows that the sequence $\{x_n\}$ is a Cauchy sequence and converges to a point $\mu\in X$ (i.e. $x_n\to\mu$ as $n\to\infty$).

Consequently, the two sub sequences $\{x_{2n+1}\}=\{T_1x_{2n}\}$ and $\{x_{2n+2}\}=\{T_2x_{2n+1}\}$ converges to the same point μ. Now, we shall show that this μ is a common fixed point for both T_1, T_2.

For this in view of the hypothesis note that

$$\|\mu - T_1\mu\|^2 = \|(\mu - x_{2n+2}) + (x_{2n+2} - T_1\mu)\|^2$$

$$\leq \|\mu - x_{2n+2}\|^2 + \|T_1\mu - T_2x_{2n+1}\|^2 + 2\|\mu - x_{2n+2}\|\|T_1\mu - T_2x_{2n+1}\|$$

$$\leq a_1 \frac{\|\mu - T_1\mu\|^2\left[1+\|x_{2n+1} - T_2x_{2n+1}\|^2\right]}{1+\|\mu - x_{2n+1}\|^2} + a_2 \frac{\|x_{2n+1} - T_2x_{2n+1}\|^2\left[1+\|x_{2n+1} - T_1\mu\|^2\right]}{1+\|\mu - x_{2n+1}\|^2}$$

$$+ a_3 \frac{\|\mu - T_2x_{2n+1}\|^2\left[1+\|x_{2n+1} - T_1\mu\|^2\right]}{1+\|\mu - x_{2n+1}\|^2} + a_4 \frac{\|\mu - x_{2n+1}\|^2\left[1+\|T_1\mu - T_2x_{2n+1}\|^2\right]}{1+\|\mu - x_{2n+1}\|^2}$$

$$+ a_5 \frac{\|\mu - T_2x_{2n+1}\|^2\left[1+\|T_1\mu - T_2x_{2n+1}\|^2\right]}{1+\|\mu - x_{2n+1}\|^2} + a_6 \frac{\|\mu - x_{2n+1}\|^2\left[1+\|\mu - T_2x_{2n+1}\|^2\right]}{1+\|\mu - x_{2n+1}\|^2}$$

$$+ a_7 \frac{\|\mu - T_1\mu\|^2 + \|x_{2n+1} - T_2x_{2n+1}\|^2 + \|\mu - x_{2n+1}\|^2}{1+\|\mu - T_1\mu\|^2\|\mu - T_2x_{2n+1}\|^2\|x_{2n+1} - T_1\mu\|^2\|\mu - x_{2n+1}\|^2} + a_8\|\mu - x_{2n+1}\|^2$$

$$+ \|\mu - x_{2n+2}\|^2 + 2\|\mu - x_{2n+2}\|\|T_1\mu - T_2x_{2n+1}\|$$

Letting $n \to \infty$, we obtained $\|\mu - T_1\mu\|^2 \leq (a_1 + a_7)\|\mu - T_1\mu\|^2$, because $a_1 + a_7 < 1$, it follows immediately that $T_1\mu = \mu$. Similarly by considering following, we get $T_2\mu = \mu$

$$\|\mu - T_2\mu\|^2 = \|(\mu - x_{2n+1}) + (x_{2n+1} - T_2\mu)\|^2$$

Next, we show that μ is a unique fixed point of T_1, T_2. Let us suppose that $v(\mu \neq v)$ is also a fixed point of T_1 and T_2. Then in view of hypothesis, we have

$$\|\mu-\nu\|^2 \le a_1 \frac{\|\mu-T_1\mu\|^2\left[1+\|\nu-T_2\nu\|^2\right]}{1+\|\mu-\nu\|^2} + a_2 \frac{\|\nu-T_2\nu\|^2\left[1+\|\nu-T_1\mu\|^2\right]}{1+\|\mu-\nu\|^2}$$

$$+ a_3 \frac{\|\mu-T_2\nu\|^2\left[1+\|\nu-T_1\mu\|^2\right]}{1+\|\mu-\nu\|^2} + a_4 \frac{\|\mu-\nu\|^2\left[1+\|T_1\mu-T_2\nu\|^2\right]}{1+\|\mu-\nu\|^2}$$

$$+ a_5 \frac{\|\mu-T_2\nu\|^2\left[1+\|T_1\mu-T_2\nu\|^2\right]}{1+\|\mu-\nu\|^2} + a_6 \frac{\|\mu-\nu\|^2\left[1+\|\mu-T_2\nu\|^2\right]}{1+\|\mu-\nu\|^2}$$

$$+ a_7 \frac{\|\mu-T_1\mu\|^2+\|\nu-T_2\nu\|^2+\|\mu-\nu\|^2}{1+\|\mu-T_1\mu\|^2\,\|\mu-T_2\nu\|^2\,\|\nu-T_1\mu\|^2\,\|\mu-\nu\|^2} + a_8\|\mu-\nu\|^2$$

$$\Rightarrow \quad \|\mu-\nu\|^2 \le \left(a_3+a_4+a_5+a_6+a_7+a_8\right)\|\nu-\mu\|^2$$

This is a contradiction for $a_3+a_4+a_5+a_6+a_7+a_8<1$ and it follows that $\mu=\nu$ and hence the common fixed point is unique.

This completes the proof of the theorem.

Theorem: 3 The two self mappings T_1, T_2 over X themselves satisfying the following condition for all $x, y \in X$ and $x \ne y$, where $a_i (i=1,2,3,....,8)$ are positive real's with $a_1+a_2+2a_3+a_4+2(a_5+a_6)+3a_7+a_8<1$ and p,q are positive integers, then T_1, T_2 have a unique common fixed point in X.

$$\|T_1^p x-T_2^q y\|^2 \le a_1 \frac{\|x-T_1^p x\|^2\left[1+\|y-T_2^q y\|^2\right]}{1+\|x-y\|^2} + a_2 \frac{\|y-T_2^q y\|^2\left[1+\|y-T_1^p x\|^2\right]}{1+\|x-y\|^2}$$

$$+ a_3 \frac{\|x-T_2^q y\|^2\left[1+\|y-T_1^p x\|^2\right]}{1+\|x-y\|^2} + a_4 \frac{\|x-y\|^2\left[1+\|T_1^p x-T_2^q y\|^2\right]}{1+\|x-y\|^2}$$

$$+ a_5 \frac{\|x-T_2^q y\|^2\left[1+\|T_1^p x-T_2^q y\|^2\right]}{1+\|x-y\|^2} + a_6 \frac{\|x-y\|^2\left[1+\|x-T_2^q y\|^2\right]}{1+\|x-y\|^2}$$

$$+ a_7 \frac{\|x-T_1^p x\|^2+\|y-T_2^q y\|^2+\|x-y\|^2}{1+\|x-T_1^p x\|^2\,\|x-T_2^q y\|^2\,\|y-T_1^p x\|^2\,\|x-y\|^2} + a_8\|x-y\|^2$$

Proof : From above Theorem-2, T_1^p and T_2^q have a unique common fixed point $\mu \in X$, so that $T_1^p \mu = \mu$ and $T_2^q \mu = \mu$.

Now $T_1^p (T_1\mu) = T_1 \left(T_1^p \mu\right) = T_1\mu$

$\Rightarrow \quad T_1\mu$ is a fixed point of T_1^p.

But μ is a unique fixed point of T_1^p.

$$\therefore \qquad T_1\mu = \mu$$

Similarly we get $T_2\mu = \mu$

$\therefore \quad \mu$ is a common fixed point of T_1 and T_2.

For uniqueness, let ν be another fixed point of T_1 and T_2, so that $T_1\nu = T_2\nu = \nu$. Then

$$\|\mu - \nu\|^2 \le a_1 \frac{\|\mu - T_1^p \mu\|^2 \left[1 + \|\nu - T_2^q \nu\|^2\right]}{1 + \|\mu - \nu\|^2} + a_2 \frac{\|\nu - T_2^q \nu\|^2 \left[1 + \|\nu - T_1^p \mu\|^2\right]}{1 + \|\mu - \nu\|^2}$$

$$+ a_3 \frac{\|\mu - T_2^q \nu\|^2 \left[1 + \|\nu - T_1^p \mu\|^2\right]}{1 + \|\mu - \nu\|^2} + a_4 \frac{\|\mu - \nu\|^2 \left[1 + \|T_1^p \mu - T_2^q \nu\|^2\right]}{1 + \|\mu - \nu\|^2}$$

$$+ a_5 \frac{\|\mu - T_2^q \nu\|^2 \left[1 + \|T_1^p \mu - T_2^q \nu\|^2\right]}{1 + \|\mu - \nu\|^2} + a_6 \frac{\|\mu - \nu\|^2 \left[1 + \|\mu - T_2^q \nu\|^2\right]}{1 + \|\mu - \nu\|^2}$$

$$+ a_7 \frac{\|\mu - T_1^p \mu\|^2 + \|\nu - T_2^q \nu\|^2 + \|\mu - \nu\|^2}{1 + \|\mu - T_1^p \mu\|^2 \|\mu - T_2^q \nu\|^2 \|\nu - T_1^p \mu\|^2 \|\mu - \nu\|^2} + a_8 \|\mu - \nu\|^2$$

$$\Rightarrow \qquad \|\mu - \nu\|^2 \le (a_3 + a_4 + a_5 + a_6 + a_7 + a_8) \|\nu - \mu\|^2$$

$$\Rightarrow \qquad \mu = \nu, \text{ since } a_3 + a_4 + a_5 + a_6 + a_7 + a_8 < 1$$

Hence μ is a unique common fixed point of T_1 and T_2 in X.

This completes the proof of the theorem.

In the last theorem we generalized the above theorems by consider a sequence of self mappings on a closed subset of a Hilbert space converges point wise to a limit mapping and show that if this limit mapping has a fixed point then this fixed point is also the limit of fixed points of the mappings of the sequence.

Theorem: 4 Let $\{T_i\}$ be a sequence of mappings of a closed subset X of a Hilbert space itself converging point wise to T and let

$$\|T_i x - T_i y\|^2 \le a_1 \frac{\|x-T_i x\|^2\left[1+\|y-T_i y\|^2\right]}{1+\|x-y\|^2} + a_2 \frac{\|y-T_i y\|^2\left[1+\|y-T_i x\|^2\right]}{1+\|x-y\|^2}$$

$$+ a_3 \frac{\|x-T_i y\|^2\left[1+\|y-T_i x\|^2\right]}{1+\|x-y\|^2} + a_4 \frac{\|x-y\|^2\left[1+\|T_i x-T_i y\|^2\right]}{1+\|x-y\|^2}$$

$$+ a_5 \frac{\|x-T_i y\|^2\left[1+\|T_i x-T_i y\|^2\right]}{1+\|x-y\|^2} + a_6 \frac{\|x-y\|^2\left[1+\|x-T_i y\|^2\right]}{1+\|x-y\|^2}$$

$$+ a_7 \frac{\|x-T_i x\|^2 + \|y-T_i y\|^2 + \|x-y\|^2}{1+\|x-T_i x\|^2\|x-T_i y\|^2\|y-T_i x\|^2\|x-y\|^2} + a_8\|x-y\|^2$$

for all $x, y \in X$ with $x \neq y$ and $a_i (i = 1,2,3,....,8)$ are non-negative real's with $a_1 + a_2 + 2a_3 + a_4 + 2(a_5 + a_6) + 3a_7 + a_8 < 1$, if each T_i has a fixed point μ_i and T has a fixed point μ, then the sequence $\{\mu_n\}$ converges to μ.

Proof : Since μ_i is a fixed point of T_i then we have

$$\|\mu-\mu_n\|^2 = \|T\mu - T_n\mu_n\|^2$$

$$= \|(T\mu - T_n\mu) + (T_n\mu - T_n\mu_n)\|^2$$

$$\le \|T\mu - T_n\mu\|^2 + \|T_n\mu - T_n\mu_n)\|^2 + 2\|T\mu - T_n\mu\|\|T_n\mu - T_n\mu_n)\|$$

$$\le a_1 \frac{\|\mu-T_n\mu\|^2\left[1+\|\mu_n-T_n\mu_n\|^2\right]}{1+\|\mu-\mu_n\|^2} + a_2 \frac{\|\mu_n-T_n\mu_n\|^2\left[1+\|\mu_n-T_n\mu\|^2\right]}{1+\|\mu-\mu_n\|^2}$$

$$+ a_3 \frac{\|\mu-T_n\mu_n\|^2\left[1+\|\mu_n-T_n\mu\|^2\right]}{1+\|\mu-\mu_n\|^2} + a_4 \frac{\|\mu-\mu_n\|^2\left[1+\|T_n\mu-T_n\mu_n\|^2\right]}{1+\|\mu-\mu_n\|^2}$$

$$+a_7 \frac{\|\mu - T_n\mu\|^2 + \|\mu_n - T_n\mu_n\|^2 + \|\mu - \mu_n\|^2}{1+\|\mu - T_n\mu\|^2 \|\mu - T_n\mu_n\|^2 \|\mu_n - T_n\mu\|^2 \|\mu - \mu_n\|^2} + a_8 \|\mu - \mu_n\|^2$$

$$+\|T\mu - T_n\mu\|^2 + 2\|T\mu - T_n\mu\|\|T_n\mu - T_n\mu_n)\|$$

Letting $n \to \infty$, so that $T_n\mu \to T\mu$, $T_n\mu_n \to \mu_n$ and $T\mu = \mu$, we get

$$\lim_{n\to\infty}\|\mu - \mu_n\|^2 \le (a_3 + a_4 + a_5 + a_6 + a_7 + a_8) \lim_{n\to\infty}\|\mu - \mu_n\|^2$$

$$\Rightarrow \lim_{n\to\infty}\|\mu - \mu_n\| = 0, \text{ since } a_3 + a_4 + a_5 + a_6 + a_7 + a_8 < 1$$

$$\Rightarrow \mu_n \to \mu \text{ as } n \to \infty$$

This completes the proof.

Conclusions

In this paper we discussed about the existence and uniqueness of a fixed point for single, a pair of mappings, positive integer’s powers of a pair mappings and a sequence of mappings defined over a closed subset of Hilbert space which is the generalization of the well known Koparde and Wghmode and Pandhare and Waghmode result in Hilbert space.

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